For Reference

NOT TO BE TAKEN FROM THIS ROOM

For Reference

NOT TO BE TAKEN FROM THIS ROOM

Ex libris
universitatis
albertaensis

UNIVERSITY OF ALBERTA
QUAECUMQUE VERA

Digitized by the Internet Archive
in 2019 with funding from
University of Alberta Libraries

https://archive.org/details/delaCruz1965

SCATTERING BY NONLOCAL POTENTIALS

by

Vicente de la Cruz

A THESIS
SUBMITTED TO THE FACULTY OF GRADUATE STUDIES
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
OF MASTER OF SCIENCE

DEPARTMENT OF PHYSICS
EDMONTON, ALBERTA
OCTOBER, 1965

UNIVERSITY OF ALBERTA
FACULTY OF GRADUATE STUDIES

The undersigned certify that they have read, and recommend to the Faculty of Graduate Studies for acceptance, a thesis entitled SCATTERING BY NONLOCAL POTENTIALS, submitted by Vicente de la Cruz, in partial fulfilment for the degree of Master of Science.

ABSTRACT

General properties, including Hermiticity, of nonlocal potentials in scattering problems are discussed. An alternative expression for flux density which simplifies interpretation of the equation of continuity is proposed. Two attempts at fitting proton-proton ^{1}S-phase shift data using potentials which have nonlocal character are made. Examples for some "equivalent" potentials are given. The Klein-Gordon equation is extended to include nonlocal interactions, and the Green function and Born approximation method are used to derive some relations of interest. Finally, the technique of second quantization is applied to Schrödinger ψ - field with nonlocal potentials; it is shown that a many-particle picture again emerges, while preserving the structure of the Schrödinger equation.

ACKNOWLEDGEMENTS

I wish to express my gratitude to Dr. M. Razavy, my supervisor, for the kindness and patience with which he guided me throughout the course of this work.

I am also indebted to Dr. B. Orman for many stimulating discussions, and to Mrs. Pauline Babet for typing the thesis.

CONTENTS

Page

Chapter I. INTRODUCTION 1

Chapter II. GENERAL PROPERTIES 4

Chapter III. S-WAVE SCATTERING (WITH NUMERICAL RESULTS) 20

Chapter IV. EQUIVALENT POTENTIALS 32

Chapter V. KLEIN-GORDON EQUATION 40

Chapter VI. SECOND QUANTIZATION 45

Bibliography 54

CHAPTER I. INTRODUCTION

The common type of potentials used in Quantum Mechanics is one consisting of those whose matrices $< \vec{r} \mid V \mid \vec{r}' >$ in the $\vec{r}$-representation are diagonal. They are known as "local" (L) potentials and form a subset of the set of "nonlocal" (NL) potentials, which are those that cannot, in general, be diagonalized in configuration space.

We do not have much physical intuitions about NL potentials. Moreover, they are mathematically complicated, Schrödinger equation having in this case the form of an integro-differential equation. One frequently has to resort to numerical computers. For these reasons, physicists have hesitated to make use of them. In the phenomenological study of nuclear forces, for example, L potentials have almost been exclusively employed.

Nevertheless, it is becoming clear that some sorts of nonlocality must be introduced. There are compelling reasons for doing so. As far back as 1936, Wheeler (1) established the equivalence of the Majorana interaction and a certain simple NL potential. More recently, extensive studies of the theory

of nuclear matter by Bethe and Goldstone and by many others(2) have shown that the optical model potential, when derived as the summation of two-body potentials, is nonlocal in character.

The optical model has been quite successful in giving a phenomenological description of elastic scattering of nucleons by nuclei. For neutrons of incident energy below 25MeV, it has been possible to find a set of optical-model parameters, independent of mass number, which vary smoothly with energy(3). This energy dependence has been explained by Perey and Buck(4) in terms of a simple nonlocality of the potential.

More specific considerations also lead one to favor the inclusion of NL interactions. For example, as Breuckner(5) pointed out, the features of the two-body interaction relevant to finite nuclei include (a) great strength of interaction leading to nonlocality, and (b) exchange character affecting nonlocality of single-particle potential. Recently, Gammel and Thaler (6) demonstrated that the proton-proton scattering data require that the singlet even parity nucleon-nucleon interaction be nonlocal.

In general, the potential must be nonlocal to represent the effect of a many-body system on the single incoming particle (7). The many-body system may be a nucleus, or the meson and baryon cloud surrounding a particle.

In passing, one may mention that nonlocality may come in in a more fundamental way, as in Field Theory(8).

* * *

We give in Chap. II a general discussion of NL potentials.

Analytic solutions for the S-wave scattering by certain NL potential-well plus local parts are given in Chap. III, together with attempts at fitting proton-proton scattering phase-shift data, for the energy range 20-340 MeV.

In most cases, the classical equivalent of a local potential can be found by inspection. With NL potentials the situation is more involved. This and other related topics will be discussed in Chap. IV, where the result of Dirac(9) is used.

For high-energy scatterings, one expects that the Klein-Gordon equation would give better results than Schrödinger equation, other things being equal. Some relevant relations for NL interactions are given in Chap. V.

In Chap. VI, the technique of second quantization is applied to Schrödinger equation with NL potentials. The process converts (as in the local case) a one-particle theory to a many-particle theory, while preserving the structure of the Schrödinger equation.

CHAPTER II. GENERAL PROPERTIES

II.1 Schrödinger Equation.

The action of a potential operator V on the state vector $|\psi>$ describing a single-particle system is represented, in the $\vec{r}$ - representation, by

$$\begin{aligned} <\vec{r}\,|\,V\,|\,\psi> &= \int d\vec{r}\,'\,<\vec{r}\,|\,V\,|\,\vec{r}\,'><\vec{r}\,'|\,\psi> \\ &= \int d\vec{r}\,'\;V(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,') \end{aligned} \qquad (II.1)$$

where

$$V(\vec{r},\vec{r}\,') = <\vec{r}\,|\,V\,|\,\vec{r}\,'>\,,$$

and

$$\psi(\vec{r}) = <\vec{r}\,|\,\psi>\,.$$

$V(\vec{r},\vec{r}\,')$ is commonly referred to as the "kernel" function.

Correspondingly, the time-independent Schrödinger equation,

$$<\vec{r}\,|\,\frac{\vec{p}^{\,2}}{2m}\,|\,\psi> + <\vec{r}\,|\,V\,|\,\psi> = E<\vec{r}\,|\,\psi>,$$

takes the form of an integro-differential equation:

$$\left(\frac{\hbar^2}{2m} \nabla^2 + E\right) \psi(\vec{r}) = \int d\vec{r}\,' \, V(\vec{r},\vec{r}\,') \, \psi(\vec{r}\,') \qquad (II.2)$$

For local (L) potentials, i.e., when $V(\vec{r},\vec{r}\,')$ has the form

$$V(\vec{r},\vec{r}\,') = v(\vec{r}) \, \delta(\vec{r}-\vec{r}\,') \qquad (II.3)$$

the integration in equation (2) becomes trivial, and one has

$$\left(\frac{\hbar^2}{2m} \nabla^2 + E\right) \psi(\vec{r}) = v(\vec{r}) \, \psi(\vec{r}).$$

II.2 Angular and Energy Dependence of Kernel Function.Hermiticity.

Certain limitations on the kernel function $V(\vec{r},\vec{r}\,')$ are found to be necessary. Clearly they must be such as to ensure the convergence of all quantities of physical interest. On physical grounds, further conditions are imposed.

Thus, for scatters having rotational symmetry, $V(\vec{r},\vec{r}\,')$ can only depend on r and r', and on the relative angle α between $\vec{r}$ and $\vec{r}\,'$, but not on the individual orientation of the two position vectors. A physically meaningful kernel can therefore be expanded in the form

$$V(\vec{r},\vec{r}\,') = \sum_{\ell} \frac{2\ell+1}{4\pi} \, \frac{K_{\ell}(r,r')}{rr'} \, P_{\ell}(\cos\alpha) \qquad (II.4)$$

where the factor $(2\ell + 1)/(4\pi\, rr')$ has been introduced for convenience. The functions $K_\ell(r,r')$ determine the angular momentum dependence of the potential.

Within Quantum Mechanics, nonlocal (NL) potentials have been used mainly in the study of nucleon-nucleus scatterings. It is of interest to see how they formally arise, and to discover some of their basic properties (Feshbach, 1958).(7) We shall be as brief as possible.

Consider a system of A + 1 nucleons, consisting of a nucleon plus a target nucleus of mass number A. The states of the target nucleus are described by a complete set of functions $\psi_i(\vec{r}_1, \ldots, \vec{r}_A)$, corresponding to energies ϵ_i. For simplicity we neglect the spin variables. The complete wave function ψ can then be expanded as follows:

$$\psi = \sum_i \psi_i\,(\vec{r}_1, \ldots, \vec{r}_A)\, u_i(\vec{r}_o) \qquad \text{(II.5)}$$

where $\vec{r}_o$ refers to the incoming nucleon. The term $i = 0$ describes the nucleus in its ground state plus a nucleon in the incident channel. Let H_T denote the total Hamiltonian:

$$H_T = H_A(\vec{r}_1, \ldots, \vec{r}_A) + T_o + V(\vec{r}_o, \vec{r}_1, \ldots, \vec{r}_A),$$

where H_A is the Hamiltonian for the A particles of the target nucleus, T_o the kinetic energy operator for the variable $\vec{r}_o$, and V the potential of the nucleon in the field of the nucleons of the target nucleus.

Substituting expansion (5) into the Schrödinger equation

$$H_T \psi = E \psi$$

where E is the total energy, and using the relation

$$H_A \psi_i = \epsilon_i \psi_i$$

as well as the orthonormal properties of the set ψ_i, we obtain a set of coupled equations:

$$(T_o + V_{ii} + \epsilon_i - E)u_i = - \sum_{j \neq i} V_{ij} u_j , \qquad \text{(II.6)}$$

where

$$V_{ij}(\vec{r}_o) = \langle \psi_i \mid V \mid \psi_j \rangle \quad .$$

To derive an uncoupled Schrödinger equation for u_o, we introduce the matrices

$$\hat{u} = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \end{pmatrix} , \qquad \text{(II.7)}$$

$$\hat{V}_o = (V_{o1}, V_{o2}, \dots) \quad , \qquad \text{(II.8)}$$

and a matrix operator $\hat{K}$ defined by

$$\hat{K}_{ij} = T_o \cdot \delta_{ij} + V_{ij} + \epsilon_i \delta_{ij} \qquad \text{(II.9)}$$

Then eq. (6) takes the form

$$(T_o + V_{oo} - E)u_o = -\hat{V}_o \hat{u}, \qquad \text{(II.10a)}$$

$$(\hat{K} - E)\hat{u} = -\hat{V}_o^{\dagger} \quad u_o , \qquad \text{(II.10.b)}$$

where $\hat{V}_o^{\dagger}$ denotes the Hermitian conjugate of $\hat{V}_o$. Solving eq. (10.b) formally we obtain

$$\hat{u} = \frac{1}{E^{(+)} - \hat{K}} \quad \hat{V}_o^{\dagger} \quad u_o \qquad \text{(II.11)}$$

where

$$E^{(+)} = E + i\eta \ , \ \eta \rightarrow 0^+ \quad . \qquad \text{(II.12)}$$

The term $i\eta$ ensures that only outgoing waves will be present in the exit channels u_i $(i \geq 1)$. Eliminating $\hat{u}$ from eq. (10.a) by means of eq. (11), we obtain

$$(T_o + V_{oo} + \widehat{V_o} \quad \frac{1}{E^{(+)} - \hat{K}} \quad \hat{V}_o^{\dagger} \quad - E)u_o = 0 \qquad \text{(II.13)}$$

The effective potential, υ , is

$$\upsilon = V_{oo} + \hat{V}_o \quad \frac{1}{E^{(+)} - \hat{K}} \quad \hat{V}_o^{\dagger} \quad . \qquad \text{(II.14)}$$

Several interesting general properties of υ emerge from a study of eq. (14). First υ is complex, due to the term $i\eta$ contained in $E^{(+)}$. The complex term arises, physically,

from processes in which the nucleon leaves the entrance channel u_o and is emitted in one of the exit channels, u_i. Of course, for this to happen, one must have $E > \epsilon_i$ so that reactions could occur.

Secondly, υ is nonlocal in character. To show this, let us insert the eigenfunctions of $\hat{K}$ into the space between the operators $\frac{1}{E^{(+)}-\hat{K}}$ and $\hat{V}_o^\dagger$ in eq. (14). In general, these eigenfunctions will consist of a discrete part and a continuum, $\{\Phi_n\}$ and $\{\Phi(\mathcal{E},\alpha)\}$, corresponding to energies $\{\mathcal{E}_n\}$ and $\{\mathcal{E}\}$, respectively; α is the degeneracy index.

We then obtain

$$\upsilon = V_{oo} + \sum_n \frac{\hat{V}_o\Phi_n><\Phi_n\hat{V}_o^\dagger}{E - \mathcal{E}_n} + \int d\alpha\int_{\epsilon_1}^{\infty} d\mathcal{E}\, \frac{\hat{V}_o\Phi(\mathcal{E},\alpha)><\Phi(\mathcal{E},\alpha)\hat{V}_o^\dagger}{E^{(+)} - \mathcal{E}} \tag{II.15}$$

Consider, for example, the second term. Writing

$$\Phi_n = \begin{pmatrix} \phi_1^{(n)} \\ \phi_2^{(n)} \\ \vdots \end{pmatrix} ,$$

the matrix element between $|\vec{r}_o>$ and $|u_o>$, of the second term above, becomes

$$\sum_n \frac{<\vec{r}_o|\hat{V}_o\Phi_n><\Phi_n\hat{V}_o^\dagger|u_o>}{E - \mathcal{E}_n}$$

$$= \sum_n \sum_{i,j\neq 0} \int \frac{V_{oi}(\vec{r}_o)\phi_i^{(n)}(\vec{r}_o)\ [\phi_j^{(n)}(\vec{r}_o{}')]^*\ V_{jo}(\vec{r}_o{}')u_o(\vec{r}_o{}')}{E - \mathcal{E}_n}\, d\vec{r}_o{}'$$

which is of the form

$$\int L(\vec{r}_o, \vec{r}_o{}') \; u_o \; (\vec{r}_o{}') \; d\vec{r}_o{}' \; ,$$

i.e., nonlocal except in extreme circumstances.

We note that the numerators in the expansions of eq. (15) are positive definite. We may decompose υ into the form

$$\upsilon = \upsilon_i + i \, \upsilon_2 \qquad \text{(II.16)}$$

where υ_1 and υ_2 are given by

$$\upsilon_1 = V_{oo} + \Sigma \frac{\hat{V}_o \Phi_n \rangle \langle \Phi_n \hat{V}_o^\dagger}{E - \mathcal{E}_n} + P \int \frac{d\mathcal{E}}{E - \mathcal{E}} \int d\alpha \; \hat{V}_o \Phi(\mathcal{E}, \alpha) \rangle \langle \Phi(\mathcal{E}, \alpha) \hat{V}_o^\dagger \; , \qquad \text{(II.17)}$$

and

$$\begin{array}{ll} \upsilon_2 = -\pi \int d\alpha \; \hat{V}_o \Phi(E, \alpha) \rangle \langle \Phi(E, \alpha) \hat{V}_o^\dagger \; , & E > \epsilon_1 \\ \quad = 0 & E < \epsilon_1 \; . \end{array} \qquad \text{(II.18)}$$

In eq. (17) the symbol P stands for Cauchy principal value. We note that υ_1 is a monotonically decreasing function of E:

$$\frac{\partial \upsilon_1}{\partial E} \leq 0 \quad ; \qquad \text{(II.19)}$$

and that υ_2 is negative definite -- as would be expected, since physically this means that the incoming nucleon density may be decreased but not increased.

Another important point which is frequently missed should also be discussed. Eqs. (17,18) show that υ_1 and υ_2 are Hermitian. If now we take the Hermitian conjugates of both sides of eq. (16), we find

$$\upsilon^{\dagger} = \upsilon_1 - i\upsilon_2 \quad . \tag{II.20}$$

Thus υ is Hermitian if and only if $\upsilon_2 = 0$ identically. The Hermitian condition

$$<\vec{r}\,|\,\upsilon\,|\,\vec{r}\,'> = <\vec{r}\,'\,|\,\upsilon\,|\,\vec{r}\,>^*$$

is therefore unduly restrictive, and in fact incompatible with the optical model.

II.3 Symmetry of Kernel function.

Within certain general formulations(10) of the nucleon-nucleus scattering problem, which encompass the highly successful optical model, one finds that it is best to deal with symmetric potentials. The configuration space is divided into two regions. The first is an internal region in which all the nucleons interact strongly and form nearly stable configurations. The second (external) region consists of channels corresponding to the various ways in which the compound nucleus can disintegrate. The dividing

surface is roughly a nuclear surface. A complete set of wave functions for the internal region are obtained by insisting that they satisfy the many-body Schrödinger equation and homogeneous boundary conditions at the dividing surface r = R. The wave function describing the actual reaction process may then be described in the internal region by an expansion in this complete set, and in the external region by the known wave functions for the disintegrated system in each channel. Thus in the internal region we need a set of functions whose radial parts are solutions of the eigenvalue equation [see eq. (III.3)].

$$\frac{d^2u_\ell^{(m)}}{dr^2} + \left[E^{(m)} - \frac{\ell(\ell+1)}{r^2}\right] u_\ell^{(m)}(r) = \int K_\ell(r,r')u_\ell^{(m)}(r')dr' .$$

For simplicity let us consider the S-wave equation

$$\frac{d^2u^{(m)}}{dr^2} + E^{(m)} u^{(m)} = \int K_o (r,r') u^{(m)} (r')dr' \qquad \text{(II.22)}$$

where $u^{(m)}(r) \equiv u_o^{(m)}(r)$.

According to the formulations of the problem (for definiteness, that of Kapur and Peierls(10)) the boundary conditions are

$$\left.\begin{aligned} u^{(m)}(0) &= 0 \\ \frac{d}{dr}[r\, u^{(m)}]_{r=R} &= ikRu^{(m)}(R) , \end{aligned}\right\} \qquad \text{(II.23)}$$

and the $u^{(m)}$ are required to be orthogonal in the following sense:

$$\int u^{(m)} u^{(n)} \, dr = 0 \quad , \; m \neq n \; . \qquad \text{(II.24)}$$

Multiplying eq. (22) by $u^{(n)}(r)$, integrate and then subtract the corresponding equation obtained by interchanging m and n, we obtain with the help of boundary conditions (23):

$$(E^{(m)} - E^{(n)}) \int u^{(n)} u^{(m)} \, dr =$$

$$\iint dr \, dr' \, K_o(r,r') [u^{(n)}(r) u^{(m)}(r') - u^{(n)}(r') u^{(m)}(r)]$$

A barely sufficient condition for the orthogonality of $u^{(m)}$ and $u^{(n)}$ when $n \neq m$ is

$$K_o(r,r') = K_o(r',r) \; ,$$

or, in general,

$$K_\ell(r,r') = K_\ell(r',r) . \qquad \text{(II.25)}$$

From eq. (4) we deduce the equivalent relation to eq. (25):

$$V(\vec{r},\vec{r}\,') = V(\vec{r}\,',\vec{r}) \qquad \text{(II.26)}$$

The above argument leading to eq. (26) is essentially that of Brown and De Dominicis$^{(11)}$.

The condition of symmetry on $V(\vec{r},\vec{r}\,')$ is met if the matrix elements $\langle \vec{r} \,|\, \upsilon_1 \,|\, \vec{r}\,' \rangle$ and $\langle \vec{r} \,|\, \upsilon_2 \,|\, \vec{r}\,' \rangle$ [eq. (16)] are real.

We note that for a complex local potential, eq. (26) is automatically satisfied. Henceforth our attention will be confined to symmetric potentials.

II.4. Flux Density.

Multiplying the Schrödinger equation (2) by $\psi^*(\vec{r})$, and then subtracting from the result the corresponding complex conjugate, we obtain

$$\nabla \cdot \left[-\frac{i\hbar}{2m} (\psi \nabla \psi^* - \psi^* \nabla \psi) \right] = \frac{2}{\hbar} \operatorname{Im} \int d\vec{r}\,' \; V(\vec{r},\vec{r}\,')\psi(\vec{r}\,')\psi^*(\vec{r}) \qquad (II.27)$$

The quantity inside the square bracket is the usual expression for probability current density (flux). Noting that the right hand quantity is in general different from zero, even for real $V(\vec{r},\vec{r}\,')$, one is tempted to modify the mathematical expression for the flux, so as to preserve the form

$$\nabla \cdot \vec{j} = 0 \qquad (II.28)$$

for real potentials, where $\vec{j}$ is the current density vector.

We recall that, for the case of a charged particle moving in a magnetic field, the expression for flux has been modified to include terms involving the vector potential, though for different reasons.

On the other hand, when the potential is complex, actual absorption occurs, as in the case of local complex potentials. We must therefore allow for the appearance of sinks.

The common practice is to leave the expression for the flux unchanged, so that eq. (27) is written

$$\nabla \cdot \vec{J} = \frac{2}{\hbar} \operatorname{Im} \int d\vec{r}\,' \; V(\vec{r},\vec{r}\,') \; \psi(\vec{r}\,') \; \psi^*(\vec{r}) \quad . \qquad \text{(II.29)}$$

For real potentials, appearance is saved by introducing the concept of "invisible channels".(12) According to this interpretation, flux sometimes leaves the elastic channel, enters the invisible channels, and reenters the elastic channel. The fact that over all flux is conserved [as can be seen from eq. (29) since $V(\vec{r},\vec{r}\,')$ is symmetric] is because as much flux departs for the invisible channels as returns from them.

It appears that an alternative, which satisfies eq. (28) when $V(\vec{r},\vec{r}\,')$ is real, and gives actual absorption when $V(\vec{r},\vec{r}\,')$ is complex, is possible. One has only to modify $\vec{J}$ in eq. (29) so as to absorb the effect of the real part of $V(\vec{r},\vec{r}\,')$. We thus obtain in place of eq. (29) the following equation of continuity:

$$\nabla \cdot \vec{J} = \frac{2}{\hbar} \operatorname{Re} \int V_2(\vec{r},\vec{r}\,') \; \psi(\vec{r}\,') \; \psi^*(\vec{r}) \; d\vec{r}\,' \qquad \text{(II.30)}$$

where

$$\vec{J} = -\frac{i\hbar}{2m}(\psi\nabla\psi^* - \psi^*\nabla\psi) - \frac{2}{\hbar n}\sum_{i=1(i)}^{n} \vec{e}\int^{x_i} dx_i \,\mathrm{Im}\int d\vec{r}\,' V_1(\vec{r},\vec{r}\,')\psi(\vec{r}\,')\psi^*(\vec{r}),$$

n = dimension of the problem $=1, 2,$ or 3,

$\vec{r} = (x_1, x_2, x_3)$

and V_1 and V_2 are the real and imaginary parts of V, respectively.

Since flux is not observable in principle, there is some arbitrariness in its mathematical definition. In the interest of simplicity of interpretation, the second alternative may, therefore, be tentatively accepted.

II.5. Wigner's Inequality.

In non-relativistic physics, the causality condition states loosely that a cause must always precede its effect. Thus, in scattering problems, the scattered waves cannot appear before the incident wave reaches the scatterer. As is well known, the condition leads to a mathematical inequality which sets for each scattering problem a lower limit for the rate of change of phase shift with respect to incident energy (Wigner's Inequality$^{(13)}$). It might appear surprising that the causality condition as stated above, being in fact a tautological statement, could have a consequence of such

physical interest. However, the causality condition is not so much a condition imposed from without as a reminder of the concepts of cause and effect already incorporated in the formulation of the problem. As such it has the status of a mathematical identity.

The Wigner's inequality has been derived for real energy-independent local potentials. Let the range of the potential be b. Then

$$\left[\frac{\partial u_\ell}{\partial k}\,\frac{\partial u_\ell{}^*}{\partial r} - u_\ell{}^*\,\frac{\partial^2 u_\ell}{\partial k \partial r}\right]_{r=b} \geq 0 \qquad \left[\begin{array}{l}\text{Wigner's}\\ \text{Inequality}\end{array}\right] . \tag{II.31}$$

where r $u_\ell(k;r)$ is the ℓ-th radial wave. For $\ell = 0$,

$$u_o(r)\Big|_{r\geq b} = \text{const x } \sin\,(kr + \delta)$$

where δ is the S-wave phase shift. Then eq. (31) gives

$$\frac{\partial\delta}{\partial k} \geq - b + \frac{\sin 2(\delta + kb)}{2k} \geq - \left(b + \frac{1}{2k}\right) .$$

We shall now show that the same relation (31) holds for real energy-independent NL potentials which vanish outside certain ranges. The following proof, first noted by Razavy(14), is a generalization of the one given by Corinaldesi and Zienau(15).

We have the equation [see eq. (III.3)]

$$u_\ell''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) u_\ell(r) = \int_0^b K_\ell(r,r')u_\ell(r')dr', \qquad r<b,$$

where b is the range of the potential. Taking the complex conjugate and the derivative with respect to k, we obtain, respectively,

$$u_\ell''^*(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) u_\ell^*(r) = \int_0^b K_\ell(r,r')\, u_\ell^*(r')dr'$$

and

$$\dot{u}_\ell''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) \dot{u}_\ell(r) + 2k\, u_\ell(r) =$$

$$\int_0^b K_\ell(r,r')\, \dot{u}_\ell(r')\, dr',$$

where the dot denotes differentiation with respect to k. We multiply the first equation by $\dot{u}_\ell(r)$, the second by $-u_\ell^*(r)$, and add up the resulting equations, obtaining

$$\dot{u}_\ell(r)u_\ell''^*(r) - u_\ell^*(r)\dot{u}_\ell''(r) - 2ku_\ell^*(r)\, u_\ell(r) =$$

$$= \int K_\ell(r,r')\, u_\ell^*(r')\dot{u}_\ell(r)dr' - \int K_\ell(r,r')\dot{u}_\ell(r')u_\ell^*(r)dr' ,$$

or

$$(\dot{u}_\ell u_\ell'^* - u_\ell^*\dot{u}_\ell')' = 2k\, u_\ell^*\, u_\ell + \int_0^b dr'\, K_\ell(r,r')[u_\ell^*(r')\dot{u}_\ell(r) - \dot{u}_\ell(r')u_\ell^*(r)]$$

If now we integrate both sides of the equation with respect to r from 0 to b, the result is

$$[\dot{u}_\ell(r)u_\ell'^{*}(r) - u_\ell^{*}(r)\dot{u}_\ell'(r)]_{r=b} = 2k \int_0^b |u_\ell|^2 \, dr \geq 0$$

since $K_\ell(r,r')$ is symmetric and $u_\ell(0) = 0$ for all k.

CHAPTER III. S-WAVE SCATTERING (WITH NUMERICAL RESULTS)

The analysis of proton-proton scattering experimental data is one of the primary means by which physicists try to understand the nature of nuclear forces. As yet, no adequate theory has been found.

In this chapter, two attempts at fitting proton-proton ^1S phase shift data, for the energy range 20 - 340 MeV, will be made. The potentials used are (in parts) nonlocal. They have the (rare) merit of being soluble, at any rate for the S-wave equation.

III.1.

Consider a system of two particles 1 and 2 interacting with each other. The total Hamiltonian operator is

$$H_T = -\frac{\hbar^2}{2m_1}\nabla_1^2 - \frac{\hbar^2}{2m_2}\nabla_2^2 + \int d(\vec{r}_1{}'-\vec{r}_2{}')\ V(\vec{r}_1-\vec{r}_2,\ \vec{r}_1{}'-\vec{r}_2{}')\Big[\qquad\Big]_{\substack{\vec{r}_1=\vec{r}_1{}' \\ \vec{r}_2=\vec{r}_2{}'}}$$

where V characterizes their interaction; the other symbols are either standard or self-explanatory. For

$$V(\vec{r}_1-\vec{r}_2,\ \vec{r}_1{}'-\vec{r}_2{}') = \upsilon(\vec{r}_1-\vec{r}_2)\ \ \delta[(\vec{r}_1-\vec{r}_2) - (\vec{r}_1{}'-\vec{r}_2{}')]$$

we have local interactions.

The motion may be separated into two parts consisting of the motion of the center of mass and their relative motion. The deduction is similar to that of the case of local interactions. The Schrödinger equation for the relative motion is

$$(\nabla^2+k^2) \quad \psi(\vec{r}) = \frac{2m}{\hbar^2} \int V(\vec{r},\vec{r}\,') \quad \psi(\vec{r}\,') \quad d\vec{r}\,' \tag{III.1}$$

where $\vec{r} = \vec{r}_1 - \vec{r}_2$

$\hbar\vec{k}$ = initial relative momentum

and m is the reduced mass.

We expand the wave function $\psi(\vec{r})$ and the kernel function $V(\vec{r},\vec{r}\,')$ in terms of spherical harmonics:

$$\psi(\vec{r}) = \sum_{\ell,m} a_\ell \frac{u_\ell(r)}{r} Y_\ell^m (\theta,\varphi) \tag{III.2}$$

$$V(\vec{r},\vec{r}\,') = \sum_\ell \frac{2\ell+1}{4\pi} \frac{K_\ell(r,r')}{rr'} P_\ell(\cos\alpha) \tag{II.4}$$

$$= \sum_{\ell,m} \frac{K_\ell(r,r')}{rr'} Y_\ell^m(\theta,\varphi) \, Y_\ell^{m*}(\theta',\varphi')$$

Substituting these expansions into eq. (1) and making use of the property

$$\int Y_{\ell'}^{m'*} (\theta',\varphi') Y_\ell^m(\theta',\varphi') d\Omega' = \delta_{\ell\ell'} \, \delta_{mm'}$$

we obtain

$$\begin{aligned} \sum_{\ell,m} \Bigg\{ & \frac{a_\ell u''_\ell(r)}{r} + \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta} \cdot \sin\theta\,\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2} \right] Y_\ell^{\,m}(\theta,\varphi)\, a_\ell \frac{u_\ell}{r^3} \\ & + k^2\, Y_\ell^{\,m}\, \frac{a_\ell u_\ell}{r} - Y_\ell^{\,m}\, \frac{2m}{\hbar^2} \int K_\ell(r,r')\, \frac{a_\ell u_\ell(r')}{r}\, dr' \Bigg\} = 0 \quad . \end{aligned}$$

We then deduce, in the usual way, the equation for the ℓ-th partial wave:

$$u''_\ell\,(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) u_\ell(r) - \frac{2m}{\hbar^2} \int K_\ell(r,r')\,u_\ell(r')\,dr' = 0 \qquad \text{(III.3)}$$

III.2

Consider a potential of range b, whose partial kernels $K_\ell(r,r')$ are solutions of the differential equation

$$\frac{d^2}{dr^2}\, K_\ell(r,r') - \left(\beta^2 + \frac{\ell(\ell+1)}{r^2}\right) K_\ell(r,r') = -\lambda\beta^2\delta(r-r'), \qquad r,r' < b \qquad \text{(III.4)}$$

with homogeneous boundary conditions

$$K_\ell(0,r') = K_\ell(b,r') = 0 \qquad \text{(III.5)}$$

This example, a special case of a more general one considered by Razavy$^{(14)}$, is instructive. Here the real constant λ in a sense (as we shall see presently) measures the strength of interaction, while $1/\beta$ (≥ 0) is an index of "nonlocality" in the sense that, as $1/\beta \to 0$, the potential becomes local. For, taking the limit $\beta \to \infty$, we have from eq. (4) that

$$K_\ell(r,r') = \lambda\delta(r-r') \quad ,$$

so that eq. (3) becomes

$$u_\ell''(r) + \left(k^2 - \frac{\ell(\ell+1)}{r^2}\right) u_\ell(r) - \frac{2m}{\hbar^2}\lambda u_\ell(r) = 0 \ , \ r < b,$$

which is, in fact, that describing a potential well of depth $-\lambda$.

Since potential well is known to be a rough approximation to the actual interaction between protons, it is worthwhile to see if this slightly more sophisticated potential (plus a small hard core) is any better. The hard core is added to give short-range repulsion.

Let us first solve the problem without the hard core. The S-wave equation is, from eq. (3)

$$u''(r) + k^2u(r) = f(r) \qquad , \quad r < b \qquad \text{(III.6)}$$

where

$$f(r) = \frac{2m}{\hbar^2} \int K_o(r,r') \, u(r') \, dr' \qquad \text{(III.7)}$$

and

$$u(r) \equiv u_o(r) \quad .$$

Multiplying on both sides of eq. (4) (for $\ell=0$) by $\frac{2m}{\hbar^2} u(r')$ and then integrating with respect to r', we get

$$f''(r) - \beta^2 f(r) = - \frac{2m}{\hbar^2} \lambda\beta^2 u(r) \qquad \text{(III.8)}$$

Eq. (8) can be used to eliminate $f(r)$ from eq. (6). The result is a fourth-order differential equation for $u(r)$:

$$\frac{d^4u}{dr^4} - (\beta^2-k^2) \frac{d^2u}{dr^2} - \beta^2\left(k^2-\frac{2m}{\hbar^2} \lambda\right) u = 0 \quad , \qquad r < b \quad \text{(III.9)}$$

The boundary conditions (5) are equivalent to

$$f(0) = f(b) = 0 \qquad \text{(III.10)}$$

In addition, we require the usual regularity condition

$$u(0) = 0 \; . \qquad \text{(III.11)}$$

Using eq. (6) we therefore have the boundary conditions

$$\begin{cases} u(0) = u''(0) = 0 \\ u''(b-) + k^2 u(b-) = 0 \end{cases} \qquad \text{(III.12)}$$

The solution of eq. (9) is

$$u = A \sinh(\mu r) + B \cosh(\mu r) + C \sin(\nu r) + D \cos(\nu r), \qquad \text{(III.13)}$$

where the constants A, ... , D are, as yet, arbitrary, and

$$\mu = \frac{1}{\sqrt{2}} \left\{ (\beta^2 - k^2) + \left[(\beta^2 - k^2)^2 + 4\beta^2 \left(k^2 - \frac{2m}{\hbar^2} \lambda \right) \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}}$$

$$\nu = \frac{1}{\sqrt{2}} \left\{ (k^2 - \beta^2) + \left[(\beta^2 - k^2)^2 + 4\beta^2 \left(k^2 - \frac{2m}{\hbar^2} \lambda \right) \right]^{\frac{1}{2}} \right\}^{\frac{1}{2}} \qquad \text{(III.14)}$$

We shall be interested in negative λ, so that the overall effect of the potential is attractive.

Outside the range of interaction,

$$u(r) = \text{const} \times \sin(kr + \delta) \qquad r > b \qquad \text{(III.15)}$$

The wavefunctions in the two regions are joined smoothly:

$$\frac{u(b-)}{u'(b-)} = \frac{u(b+)}{u'(b+)} \qquad \text{(III.16)}$$

Three of the constants in eq. (13) can be eliminated by the boundary conditions (12). Then eq. (16) gives the desired relation for the phase shift δ as a function of k:

$$k \cot [\delta+kb] = \frac{1}{\mu^2+\nu^2} [(\mu^2+k^2)\nu \cot(\nu b)+(\nu^2-k^2)\mu \coth(\mu b)] \tag{III.17}$$

If a hard core of radius $c < b$ is added, condition (11) must be replaced by

$$u(r) = 0, \quad r \leq c \tag{III.18}$$

Let us also modify the boundary conditions (5) for K_ℓ, to read

$$K_o(c,r') = K_o(b,r') = 0 \tag{III.19}$$

It follows that the boundary conditions for $u(r)$ in the region $c \leq r \leq b$ are obtained from (12) by replacing $u(0)$ and $u''(0)$ by $u(c)$ and $u''(c)$, respectively. We obtain an equation slightly different from eq. (17), viz.

$$k \cot (kb+\delta) = \frac{1}{\mu^2+\nu^2} \left\{(\mu^2+k^2)\nu \cot [\nu(b-c)]+(\nu^2-k^2)\mu \coth [\mu(b-c)]\right\} \tag{III.20}$$

With eq. (20), we try to fit the ^1S phase shift to Breit's values (YLAM)(16), by adjusting the various parameters. We find that the following set of values, though perhaps not the best choice, gives a reasonable fit. These values are close to those of Giltinan and Thaler(17).

$\lambda = -27$ MeV, $\beta = 2.174$ F^{-1}, $b = 2.6$ F, $c = 0.157$ F.

The result is given in table I, under the potential υ_I.

E_{lab}(MeV)	20	100	180	260	340
δ (υ_I) (rad.)	0.884	0.300	0.038	-0.090	-0.157
δ (υ_{II})(rad.)	0.857	0.341	0.094	-0.071	-0.195
δ(YLAM)(rad.)	0.856	0.380	0.136	-0.033	-0.195

Table I.

III.3.

Consider another potential, consisting of a nonlocal one of range b given by eq. (4), and the Eckart potential

$$V_E = - \frac{2\ V_o \gamma^2 e^{-\gamma r}}{(1+V_o e^{-\gamma r})^2}$$

for $r > b$.

We shall confine the parameters β and λ to values satisfying the inequalities

$$\begin{gathered} 4\beta^2\left(\frac{2m}{\hbar^2}\lambda - k^2\right) > (\beta^2 - k^2)^2 \\ \beta^2 > k^2 \end{gathered} \tag{III.21}$$

for the energy range 20 - 340 MeV. This means, in particular, that we want the nonlocal potential to be repulsive.

For $r < b$, the differential equation is identical with eq. (9). Its solutions are linear combination of exponentials $e^{\alpha r}$ where α are roots of the algebraic equations

$$2\alpha^2 = (\beta^2-k^2) \pm i \sqrt{4\beta^2(\frac{2m}{\hbar^2} \lambda - k^2) - (\beta^2-k^2)^2}$$

The α's are given by

$$\alpha = \rho \pm i\omega, \quad -\rho \pm i\omega \tag{III.22a}$$

where

$$\begin{aligned} \rho &= \frac{1}{2}\left\{2\beta\sqrt{\frac{2m\ \lambda}{\hbar^2} - k^2} + (\beta^2 - k^2)\right\}^{\frac{1}{2}} \\ \omega &= \frac{1}{2}\left\{2\beta\sqrt{\frac{2m\ \lambda}{\hbar^2} - k^2} - (\beta^2 - k^2)\right\}^{\frac{1}{2}} \end{aligned} \tag{III.22b}$$

Next we employ the boundary conditions (12) to eliminate three of the four constants appearing in u(r). The result is

$$u(r)_{r<b} = \frac{\sinh\ (\rho r)\ \cos\ (\omega r)}{(\rho^2-\omega^2+k^2)\sinh(\rho b)\cos(\omega b)\ -\ 2\rho\omega\ \cosh(\rho b)\ \sin(\omega b)}$$

$$-\ \frac{\cosh\ (\rho r)\ \sin\ (\omega r)}{(\rho^2-\omega^2+k^2)\ \cosh(\rho b)\ \sin(\omega b)\ +\ 2\rho\omega\ \sinh(\rho b)\ \cos(\omega b)}$$

where we have set the remaining constant equal to unity, as we are mainly interested in the logarithmic derivative. The value of this latter quantity at r = b is then found to be

$$F \equiv \frac{u'(b-)}{u(b-)}$$

$$= \frac{\rho(\rho^2+\omega^2+k^2)\big(\coth(\rho b)-\tanh(\rho b)\big)+\omega(\rho^2+\omega^2-k^2)\big(\cot(\omega b)+\tan(\omega b)\big)}{2\rho\omega\left\{\frac{\cot(\omega b)}{\coth(\rho b)} + \frac{\coth\ (\rho b)}{\cot(\omega b)}\right\}} \tag{III.23}$$

For $r > b$, the differential equation is

$$u''(r) + k^2 u(r) = -\frac{2 V_o \gamma^2 e^{-\gamma r}}{(1+ V_o e^{-\gamma r})^2} u(r) \qquad \text{(III.24)}$$

This has solutions of the form(20)

$$u(r)_{r>b} = C\, g(r) + D\, h(r) \qquad \text{(III.25)}$$

where C and D are arbitrary constants,

$$\left\{ \begin{aligned} g(r) &= \left\{ 1 + \frac{Y(r)}{1+4 \frac{k^2}{\gamma^2}} \right\} \sin kr + \frac{2\, k/\gamma}{1+4 \frac{k^2}{\gamma^2}} Y(r) \cos kr \\ h(r) &= \left\{ 1 + \frac{Y(r)}{1+4 \frac{k^2}{\gamma^2}} \right\} \cos kr - \frac{2\, k/\gamma}{1+4 \frac{k^2}{\gamma^2}} Y(r) \sin kr \end{aligned} \right. \qquad \text{(III.26)}$$

and

$$Y(r) = -\frac{2 V_o e^{-\gamma r}}{1 + V_o e^{-\gamma r}} \qquad \text{(III.27)}$$

As $r \rightarrow \infty$, $Y(r) \rightarrow 0$ and so $u(r)$ has the required asymptotic form

$$u(r) \sim C \sin kr + D \cos kr.$$

Hence

$$\frac{D}{C} = \tan \delta \quad . \qquad \text{(III.28)}$$

The logarithmic derivative is

$$\frac{u'(b+)}{u(b+)} = \frac{C\ g'(b) + D\ h'(b)}{C\ g(b) + D\ h\ (b)} = \frac{g'(b) + \tan\delta \cdot h'(b)}{g\ (b) + \tan\delta \cdot h\ (b)}$$

The logarithmic derivative must be continuous at r = b. In terms of $F = \frac{u'(b-)}{u\ (b-)}$ as defined in eq. (23), we therefore obtain

$$\tan\delta = \frac{g'(b) - F\ g(b)}{Fh(b) - h'\ (b)}\ ,$$

or else,

$$\tan\ (kb + \delta) = \frac{\tan kb + \tan\delta}{1 - \tan kb \cdot \tan\delta}$$

$$= \frac{\{F \sin(kb)h(b) - \cos(kb)\cdot g(b)\} + \{g'(b)\cos kb - h'(b) \sin kb\}}{\{F \cos(kb)h(b) + \sin(kb)\cdot g(b)\} - \{g'(b)\sin kb + h'(b) \cos kb\}}$$

Making use of eq. (26) defining h(r) and g(r), we finally obtain

$$\tan\ (kb + \delta) = \frac{K_3 - K_2F}{K_1F - K_4}\ , \tag{III.29}$$

where

$$\left.\begin{aligned} K_1 &= 1 + \frac{1}{1 + 4\,\frac{k^2}{\gamma^2}}\ Y(b) \\ K_2 &= \frac{2\ k/\gamma}{1 + 4\,\frac{k^2}{\gamma^2}}\ Y(b) \end{aligned}\right\} \tag{III.30}$$

$$\left.\begin{aligned} K_3 &= k\left[1 + \frac{1}{1 + 4\,\frac{k^2}{\gamma^2}\;Y(b)}\right] + \frac{2\;k/\gamma}{1+4\,\frac{k^2}{\gamma^2}}\;Y\,(b) \\ K_4 &= \frac{Y'(b) - 2\,\frac{k}{\gamma}\,Y(b)}{1 + 4\,\frac{k^2}{\gamma^2}} \end{aligned}\right\} \qquad (III.30)$$

and $Y(r)$ is as defined in eq. (27).

With eq. (29) we try to fit the ^1S phase shifts to Breit's values (YLAM), by adjusting the various parameters, subject to the restriction (21). We find that the following set of values gives a reasonable fit:

$$\lambda = 788 \text{ MeV}, \quad \beta = 4.3 \text{ F}^{-1}, \quad b = 0.7 \text{ F}$$

$$V_o = 73 \text{ MeV}, \quad \gamma = 2 \text{ F}^{-1} .$$

The result is given in table I, under the potential υ_{II}.

CHAPTER IV. EQUIVALENT POTENTIALS

IV.1 Momentum-dependent Potential.

Nonlocal potentials are mathematically equivalent to momentum-dependent potentials(1). That is to say, for any NL potential operator V characterized by its matrix elements $V(\vec{r},\vec{r}\,') \equiv \langle\vec{r}\,|\,V\,|\,\vec{r}\,'\rangle$, a function $U(\vec{r},\vec{p})$ of the position and momentum operators, $\vec{r}$ and $\vec{p}$, can be found such that, formally, the following holds:

$$U(\vec{r},\vec{p})\,\psi(\vec{r}) = \int V(\vec{r},\vec{r}\,')\,\psi\,(\vec{r}\,')\,d\vec{r}\,' \qquad \text{(IV.1)}$$

For we can expand the wave function $\psi(\vec{r}\,')$ appearing in the integral about the point $\vec{r}\,' = \vec{r}$, thus,

$$\begin{aligned}\psi(\vec{r}\,') &= \psi(\vec{r} + \vec{r}\,' - \vec{r})\\ &= \Sigma\,\frac{1}{n!}\,[(\vec{r}\,' - \vec{r})\cdot\nabla]^n\,\psi(\vec{r})\\ &= \Sigma\,\frac{1}{n!}\,[(\vec{r}\,' - \vec{r})\cdot i\frac{\vec{p}}{\hbar}\,]^n\,\psi(\vec{r})\\ &= e^{\frac{i}{\hbar}(\vec{r}\,'-\vec{r})\cdot\vec{p}}\,\psi(\vec{r})\end{aligned}$$

Therefore

$$U(\vec{r},\vec{p}) = \int d\vec{r}\,'\,V(\vec{r},\vec{r}\,')\,e^{\frac{i}{\hbar}(\vec{r}\,'-\vec{r})\cdot\vec{p}} \qquad \text{(IV.2)}$$

As an example, consider a potential V_M, having Majorana exchange character, defined by

$$\langle\vec{r} \,|\, V_M \,|\, \psi\rangle = V_M(\vec{r})\ \psi(-\vec{r}) \quad .$$

Since $\langle\vec{r} \,|\, V_M \,|\, \psi\rangle = \int d\vec{r}\,'\ \langle\vec{r} \,|\, V_M \,|\, \vec{r}\,'\rangle \langle\vec{r}\,' \,|\, \psi\rangle$,
it follows that $\langle\vec{r} \,|\, V_M \,|\, \vec{r}\,'\rangle = V_M(\vec{r})\ \delta(\vec{r} + \vec{r}\,')$.

Using eq. (2), we find the equivalent momentum-dependent potential to be

$$U_M(\vec{r},\vec{p}) = V_M(\vec{r})\ e^{-\frac{i}{\hbar}\,2\vec{r}\cdot\vec{p}} \quad .$$

The example is due to Wheeler(1).

IV.2. Local Static Potentials in Born Approximation.

Equivalent field-theoretical nuclear potentials are defined as those which give the same K-matrix elements in the first Born approximation(19). In the non-relativistic reduction they may be taken over and used in the Schrödinger equation.

Given a NL potential whose S-wave kernel $K(r,r')$ (eq. III.3) with $\ell = 0$) is energy-independent, we wish here to find a local static potential which gives the same S-phase shift in Born approximation.

The phase shifts, in Born approximation, are given by

$$\tan \delta_L = -\frac{2m}{\hbar^2 k} \int_0^\infty dr \sin^2 kr\ \upsilon(r) \qquad \text{(IV.3)}$$

and

$$\tan \delta_{NL} = - \frac{2m}{\hbar^2 k} \int_0^\infty dr_1 \sin kr_1 \int_0^\infty K(r_1,r_2) \sin kr_2 dr_2 \qquad (IV.4)$$

Eq. (4) is easily derived by converting eq. (III.3) for $\ell = 0$ into the following integral equation:

$$u(r) = \sin kr - \frac{2m}{\hbar^2} \int_0^\infty G(r,r') \int_0^\infty K(r',r'')u(r'')dr'dr''$$

where

$$G(r,r') = \frac{1}{k} \sin kr_< \cos kr_> .$$

Here $r_<$ denotes the lesser of r and r '. The first Born approximation is

$$u(r)_B = \sin kr - \frac{2m}{\hbar^2} \int_0^\infty G(r,r') \int_0^\infty K(r',r') \sin kr' dr'dr'' .$$

Its asymptotic expression

$$u(r)_B \sim \sin kr - \cos kr \cdot \frac{2m}{\hbar^2} \cdot \frac{1}{k} \int_0^\infty dr' \sin kr \int_0^\infty K(r',r'') \sin kr'' dr''$$

implies eq. (4).

We ask the question: given that

$$\int_0^\infty dr_1 \sin^2 kr_1 \upsilon(r_1) = \int_0^\infty dr_1 \sin kr_1 \int_0^\infty K(r_1,r_2) \sin kr_2 dr_2 \qquad (IV.5)$$

for all k (and hence for large k), what is the functional f(K) where $f(K) = \upsilon$?

First we note that condition (5) is equivalent to

$$\frac{\partial}{\partial k}\int_0^\infty dr_1\ \sin^2 kr_1\upsilon(r_1) = \frac{\partial}{\partial k}\int_0^\infty dr_1 \sin kr_1 \int_0^\infty K(r_1,r_2)\sin kr_2 dr_2 \tag{IV.6}$$

for all k since both sides of equation (5) vanish for k = 0.

The left hand side of eq. (6) gives

$$\frac{\partial}{\partial k}\int_0^\infty dr\ \sin^2 kr\ \upsilon(r) = \int_0^\infty dr\cdot 2\ \sin kr\cdot\cos kr\cdot r\ \upsilon(r)$$

$$= \int_0^\infty dr\cdot\sin 2\ kr\ r\ \upsilon(r)$$

$$= \frac{1}{4}\int_0^\infty dr\cdot\sin kr\ r\ \upsilon(\tfrac{r}{2})$$

$$= \text{sine transform of } \frac{1}{4}\ r\ \upsilon(\tfrac{r}{2}) \quad .$$

The left hand side of eq. (6) gives, after some deductions,

$$\frac{\partial}{\partial k}\int_0^\infty dr_1 \sin kr_1 \int_0^\infty K(r_1,r_2)\ \sin kr_2\ dr_2$$

$$= \text{sine transform of } \frac{1}{2}r\int_0^r dr'\ K(r-r',r') - r\int_0^\infty dr'\ K(r+r',r')$$

Hence, taking the inverse sine transforms, we obtain

$$\frac{1}{4}\ r\upsilon(\tfrac{r}{2}) = \frac{1}{2}\ r\int_0^r dr'\ K(r-r',r') - r\int_0^\infty dr'\ K(r'+r,r')$$

almost everywhere, or

$$\upsilon(r) = 2 \int_0^{2r} dr' \; K\,(2r-r', \; r') \; - \; 4 \int_0^{\infty} dr' \; K(2r+r',r') \qquad \text{(IV.7)}$$

As an example, let

$$K(r',r'') = -e^{-(r'+r'')} \quad .$$

Using eq. (7), we get

$$\upsilon(r) = 2\, e^{-2r}(1 - 2r) \; .$$

Equation (5) is easily verified.

IV.3.

In a situation where the fact that $\vec{p}$ and $\vec{r}$ do not commute may be neglected, thereby reducing the description to a classical one, it is natural to ask for the (in some sense) classical equivalent of the potential operator V. We may take $U(\vec{r},\vec{p})$ as given in eq. (2) to be such a one, treating $\vec{p}$ in the integrand as a C-number variable; or else, following Dirac$^{(9)}$, we may again choose $U(\vec{r},\vec{p})$ but with $\vec{r}$ defined as the mean of the position variables appearing in the integral, thus:

$$U_{cl}\,(\vec{r},\vec{p}) = \int V(\vec{r}\,',\vec{r}\,'')e^{-i(\vec{r}\,'-\vec{r}\,'')p/\hbar} d(\vec{r}\,'-\vec{r}\,'') \qquad \text{(IV.8)}$$

where

$$\vec{r} = \frac{1}{2}\,(\vec{r}\,' + \vec{r}\,'') \qquad .$$

Let us consider a potential whose partial kernel functions $K_\ell(r,r')$ (eq. II.4) are solutions of the differential equation

$$\frac{d^2}{dr^2}\,[K_\ell(r,r')\,\upsilon^{\frac{1}{2}}(r)] - \left(\beta^2/\hbar^2 + \frac{\ell(\ell+1)}{r^2}\right)\left(K_\ell(r,r')\,\upsilon^{\frac{1}{2}}(r)\right)$$

$$= -\lambda\ \beta^2/\hbar^2\,\upsilon^{\frac{1}{2}}\,(r')\ \delta(r-r') \qquad \text{(IV.9)}$$

with boundary conditions

$$K_\ell(0,r') = K_\ell(\infty,r') = 0 \qquad \text{(IV.10)}$$

This potential has the property that, as $\beta \to \infty$, it reduces to a local potential $\lambda\upsilon(r)$. The solution of equ. (9) is found to be (14)

$$K_\ell(r,r') = -\,\lambda\frac{\beta^3}{\hbar^3}\ rr'\ \upsilon^{\frac{1}{2}}(r)\ \upsilon^{\frac{1}{2}}(r')\ j_\ell(i\frac{\beta}{\hbar}r_<)\,h_\ell^{(1)}(i\frac{\beta}{\hbar}\,r_>) \qquad \text{(IV.11)}$$

where j_ℓ and $h_\ell^{(1)}$ are spherical Bessel and Hankel functions of the first kind, respectively.

Then

$$V(\vec{r},\vec{r}\,') = -\frac{1}{4\pi}\,\frac{1}{rr'\hbar^3}\,\lambda\beta^3\; rr'\;\upsilon^{\frac{1}{2}}(r)\upsilon^{\frac{1}{2}}(r')\sum_{\ell}(2\ell+1)j_{\ell}(i\tfrac{\beta}{\hbar}r_<)\,h_{\ell}^{(1)}(i\tfrac{\beta}{\hbar}\,r_>)P_{\ell}(\cos\,\alpha)$$

$$= -\frac{\lambda\beta^3}{4\pi\hbar^3}\;\upsilon^{\frac{1}{2}}(r)\upsilon^{\frac{1}{2}}(r')\left[-\frac{e^{-\frac{\beta}{\hbar}|\vec{r}-\vec{r}\,'|}}{\frac{\beta}{\hbar}\;|\vec{r}-\vec{r}\,'|}\right]$$

$$= \frac{1}{4\pi}\,\frac{\lambda}{\hbar^2}\cdot\beta^2\;\upsilon^{\frac{1}{2}}(r)\upsilon^{\frac{1}{2}}(r')\,\frac{e^{-\frac{\beta}{\hbar}|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \qquad \text{(IV.10)}$$

Let us set $\upsilon(r) = e^{-r/a}$. From eq. (8),

$$U_{cl}(\vec{r},\vec{p}) = \int V(\vec{r}\,',\vec{r}\,'')\,e^{-i(\vec{r}\,'-\vec{r}\,'')\vec{p}/\hbar}d(\vec{r}\,'-\vec{r}\,'')$$

$$= \frac{\lambda}{4\pi}\,\frac{\beta^2}{\hbar^2}\int e^{-\frac{r'+r''}{2a}}\,\frac{e^{-\frac{\beta}{\hbar}|\vec{r}\,'-\vec{r}\,''|}}{|\vec{r}\,'-\vec{r}\,''|}\,e^{-i(\vec{r}\,'-\vec{r}\,'')\vec{p}/\hbar}d(\vec{r}\,'-\vec{r}\,'') \quad .$$

Since the factor $\dfrac{e^{-\frac{\beta}{\hbar}|\vec{r}\,'-\vec{r}\,''|}}{|\vec{r}\,'-\vec{r}\,''|}$ ensures that the major contribution to the integral comes from the region in which $\vec{r}\,' \simeq \vec{r}\,''$, the effect of replacing $\vec{r}\,'+\vec{r}\,''$ by $|\vec{r}\,'+\vec{r}\,''|$ should be small. This is done to obtain a closed form, thus making the result physically more transparent. Then

$$U_{cl}(\vec{r},\vec{p}) = \frac{\lambda}{4\pi}\frac{\beta^2}{\hbar^2}\, e^{-r/a} \int \frac{e^{-\frac{\beta}{\hbar}p}}{\rho}\, e^{-i\vec{\rho}\cdot\vec{p}/\hbar}\, d\vec{\rho} \;, \quad \vec{\rho} = \vec{r}\,' - \vec{r}\,''$$

$$= \frac{\lambda}{4\pi}\frac{\beta^2}{\hbar^2}\, e^{-r/a} \int d\rho\cdot\rho\; e^{-\frac{\beta}{\hbar}p}\; 2\pi\int_0^{\pi} e^{-i\rho\frac{p}{\hbar}\cos\theta} \sin\theta\, d\theta$$

$$= \lambda\,\frac{\beta^2}{\hbar p}\; e^{-r/a} \int_0^{\infty} d\rho\; e^{-\frac{\beta}{\hbar}p}\; \sin(\tfrac{p}{\hbar}\rho)$$

$$= \lambda\,\beta^2\,\frac{e^{-r/a}}{\beta^2 + p^2}$$

We observe that, as $\beta \to \infty$,

$$U_{cl}\,(\vec{r},\vec{p}) \to \lambda\; e^{-r/a} = \lambda\upsilon(r),$$

as should be expected.

CHAPTER V. KLEIN-GORDON EQUATION

V.1

We shall take as our starting point the Klein-Gordon equation in a local field $V(\vec{r})$:

$$(\nabla^2 - m^2)\psi(\vec{r}) + (E - V(\vec{r}))^2 \psi(\vec{r}) = 0 \qquad (V.1)$$

where E is the relativistic energy of the one-particle system. It will be taken for granted that eq. (1) represents a kinematical improvement over the usual Schrödinger equation, as a description of high-energy scattering processes. A discussion of the many difficulties inherent in eq. (1) lies outside the scope of the present work$^{(21)}$. Our object is to extend eq. (1) to include nonlocal interactions, and from there to derive some relations of interest.

The transition to the case of nonlocal interactions is equivalent to adopting a momentum-dependent potential $U(\vec{r},\vec{p})$ in place of $V(\vec{r})$, since formally

$$U(\vec{r},-i\nabla)\psi(\vec{r}) = \int V(\vec{r},\vec{r}\,')\ \psi(\vec{r}\,')\ d\vec{r}\,' , \qquad (V.2)$$

or, more generally,

$$U(\vec{r},-i\nabla)f(\vec{r}) = \int V(\vec{r},\vec{r}\,')\ f(\vec{r}\,')d\vec{r}\,', \qquad (V.3)$$

where the function $V(\vec{r},\vec{r}\,')$ is determined by $U(\vec{r},\vec{p})$, and $f(\vec{r})$ is a function of $\vec{r}$. (See Chapter IV).

Then eq. (1) becomes

$$(\nabla^2-m^2)\,\psi(\vec{r}) + (E - U(\vec{r},-i\nabla))^2\,\psi(\vec{r}) = 0$$

or else, making use of eq. (3),

$$\nabla^2\psi(\vec{r}) + (E^2-m^2)\,\psi(\vec{r}) = \int d\vec{r}\,'\;L(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,') \qquad (V.4)$$

where

$$L(\vec{r},\vec{r}\,') = 2E\,V(\vec{r},\vec{r}\,') - \int d\vec{r}\,''\;V(\vec{r},\vec{r}\,'')\,V(\vec{r}\,'',\vec{r}\,') \qquad (V.5)$$

We note that $L(\vec{r},\vec{r}\,')$ is symmetric, since $V(\vec{r},\vec{r}\,')$ is assumed to be so. (Chapter II).

Eq. (4) may be rewritten in the form of an integral equation. Let $\vec{k}$ denote the incident wave vector,

$$k = \sqrt{E^2 - m^2},$$

and let

$$F(\vec{r}) = \int d\vec{r}\,'\;L(\vec{r},\vec{r}\,')\,\psi(\vec{r}\,') \qquad (V.6)$$

Then eq. (4) becomes

$$(\nabla^2 + k^2)\,\psi\,(\vec{r}) = F(\vec{r}) \quad . \qquad (V.7)$$

Since the Green's function,

$$G(\vec{r},\vec{r}\,') = -\frac{1}{4\pi}\frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \tag{V.8}$$

satisfies the equation

$$(\nabla^2 + k^2)\, G(\vec{r},\vec{r}\,') = \delta(\vec{r}-\vec{r}\,') \quad , \tag{V.9}$$

it follows immediately that

$$\begin{aligned}\psi(\vec{r}) &= \psi_o(\vec{r}) + \int G(\vec{r},\vec{r}\,')\, F(\vec{r}\,')\, d\vec{r}\,' \\ &= \psi_o(\vec{r}) - \frac{1}{4\pi}\int d\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\int d\vec{r}\,''\, L(\vec{r}\,',\vec{r}\,'')\psi(\vec{r}\,'')\end{aligned} \tag{V.10}$$

is a solution of eq. (4), where $\psi_o(\vec{r})$ is any solution of the associated homogeneous equation:

$$\psi_o(\vec{r}) = C_1\, e^{i\vec{k}\cdot\vec{r}} + C_2\, e^{-i\vec{k}\cdot\vec{r}} \quad .$$

For scattering problems, we take $C_1 = 1$, $C_2 = 0$, so as to obtain the correct asymptotic form. Thus

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi}\int d\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}\int d\vec{r}\,''\, L(\vec{r}\,',\vec{r}\,'')\, \psi(\vec{r}\,''). \tag{V.11}$$

The integral equation (11) may be solved by iteration. The first Born approximation is

$$\psi(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} - \frac{1}{4\pi} \int d\vec{r}\,' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} \int d\vec{r}\,'' L(\vec{r}\,',\vec{r}\,'')e^{i\vec{k}\cdot\vec{r}\,''}$$

$$= e^{i\vec{k}\cdot\vec{r}} - \frac{E}{2\pi} \int d\vec{r}\,'d\vec{r}\,'' \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,',\vec{r}\,'')\, e^{i\vec{k}\cdot\vec{r}\,''}$$

$$+ \frac{1}{4\pi} \int d\vec{r}\,'\, d\vec{r}\,''\, d\vec{r}_1 \frac{e^{ik|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|} V(\vec{r}\,',\vec{r}_1)V(\vec{r}_1,\vec{r}\,'')e^{i\vec{k}\cdot\vec{r}\,''} \qquad (V.12)$$

The Born approximation has been tested for complex potential well (eq.(III.4) with λ complex) at high incident energies (~ 1 BeV), by calculating both the exact and approximate S-wave scattering cross-sections. It was found to be remarkably good, the error being within 3%.

V.3.

The Klein-Gordon equation (4) can easily be decomposed into partial-wave equations, in a way completely analogous to that of section III.1. Thus writing

$$\psi(\vec{r}) = \sum_{\ell,m} a_\ell \frac{u_\ell(r)}{r} Y_\ell^m(\theta,\varphi) \qquad (V.13)$$

$$V(\vec{r},\vec{r}\,') = \sum_\ell \frac{2\ell+1}{4\pi} \frac{K_\ell(r,r')}{rr'} P_\ell(\cos\alpha) \qquad (V.14)$$

whence

$$L(\vec{r},\vec{r}\,') = \sum_{\ell} \frac{2\ell+1}{4\pi} L_{\ell} \frac{(r,r')}{rr'} P_{\ell} (\cos \alpha) \qquad (V.15)$$

where

$$L_{\ell}(r,r') = 2E\, K_{\ell}(r,r') - \int dr''\, K_{\ell}(r\ ,r'')K_{\ell}(r',r'') \qquad (V.16)$$

we obtain for the ℓ-th partial wave the equation

$$[u_{\ell}'' + k^2 - \frac{\ell(\ell+1)}{r^2}]\ \ u_{\ell} = \int L_{\ell}\ (r,r')\ u_{\ell}(r')\ dr'. \qquad (V.17)$$

CHAPTER VI. SECOND QUANTIZATION

The general procedure for the quantization of the equations of motion of a classical system may be applied to a wave field. In this way, for example, one obtains the theory of photons from the classical electromagnetic field.

The quantization of a ψ-field such as that described by the Schrödinger equation is commonly known as second quantization. The transition from classical particle mechanics to wave mechanics constitutes the first quantization. It has been shown$^{(18)}$ that, for the case of local interactions, the second quantization of a Schrödinger ψ-field converts a one-particle theory to a many-particle theory. We shall generalize the argument to include nonlocal interactions as well. For simplicity, we treat the one-dimensional case. The presentation closely follows that of Schiff$^{(18)}$.

VI.1 First, we postulate the existence of a Lagrangian density of the form

$$\mathcal{L} = i\hbar\, \psi^* \dot{\psi} - \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x} \frac{\partial \psi^*}{\partial x} - \psi^*(x) \int_{-\infty}^{\infty} K(x,y)\psi(y)dy \qquad \text{(VI.1)}$$

where dot denotes differentiation with respect to time, and K(x,y) is the kernel of the nonlocal potential.

The Action A of the system between times t_1 and t_2 is defined in terms of the Lagrangian density as

$$A = \int_{t_1}^{t_2} \int_{-\infty}^{\infty} \mathcal{L} \, dx \, dt \qquad \text{(VI.2)}$$

We vary the Action A with respect to ψ and ψ^* independently, subject to the usual restrictions

$$\delta A(t_1) = \delta A(t_2) = 0 \quad , \qquad \text{(VI.3)}$$

and employ the Principle of Least Action to derive the equations of motion.

In varying with respect to ψ^*, the functional dependence of $\mathcal{L}$ is

$$\mathcal{L} = \mathcal{L}\left(\psi^*, \frac{\partial \psi^*}{\partial x} \right) \quad ,$$

and so the Euler's equation would result:

$$\frac{\partial \mathcal{L}}{\partial \psi^*} - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial \left(\frac{\partial \psi^*}{\partial x}\right)} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial \dot{\psi}^*} = 0 \qquad \text{(VI.4)}$$

Substituting the expression for $\mathcal{L}$ (eq. 1) into eq. (4), we obtain

$$i\hbar\dot{\psi} - \int_{-\infty}^{\infty} K(x,y)\, \psi(y)\, dy - \frac{\partial}{\partial x} \left(- \frac{\hbar^2}{2m} \frac{\partial \psi}{\partial x} \right) = 0 \quad ,$$

or

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{dx^2} + \int K(x,y)\,\psi(y)\,dy \qquad \text{(VI.5)}$$

which is the time-dependent Schrödinger equation.

In varying with respect to ψ, we first note that the Lagrangian density

$$\mathcal{L}' = i\hbar\,\psi^*\dot{\psi} - \frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} - \psi(x)\int K(x,y)\,\psi^*(y)\,dy$$

which can be obtained from $\mathcal{L}$ by interchanging ψ and ψ^* in the last term, gives the same action as does $\mathcal{L}$, since $K(x,y)$ must be symmetric. Hence again we can use the Euler's equation directly. We obtain an equation conjugate to eq. (5).

$$-i\hbar\dot{\psi}^* = -\frac{\hbar^2}{2m}\frac{\partial^2\psi^*}{dx^2} + \int K(x,y)\,\psi^*(y)\,dy\ . \qquad \text{(VI.6)}$$

The same equations (5,6) will be obtained if, instead of ψ and ψ^*, one treats ψ_1 and ψ_2 as the independent fields, where ψ_1 and ψ_2 are defined by

$$\psi = \psi_1 + i\psi_2\ .$$

VI.2 The Hamiltonian density of the system is defined to be

$$\mathcal{H}\ (x,t) = \pi\dot{\psi} + \pi^{(*)}\dot{\psi}^* - \mathcal{L} \tag{VI.7}$$

where

$$\pi = \frac{\partial\mathcal{L}}{\partial\dot{\psi}} = i\hbar\psi^* \tag{VI.8}$$

$$\pi^{(*)} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}^*} = 0\ . \tag{VI.9}$$

Hence

$$\mathcal{H} = i\hbar\psi^*\dot{\psi} + 0 - i\hbar\psi^*\dot{\psi} + \frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} + \psi^*(x)\int K(x,y)\psi(y)dy$$

$$= \frac{\hbar^2}{2m}\frac{\partial\psi}{\partial x}\frac{\partial\psi^*}{\partial x} + \psi^*(x)\int K(x,y)\ \psi(y)\ dy \tag{VI.10}$$

Note that $\pi^{(*)}$ and ψ^* cannot be regarded as a pair of canonically conjugate variables since they cannot be made to satisfy the usual Poisson equation. This gives no problem since they can be eliminated from the Hamiltonian density altogether. Thus, using eqs. (8,9), one obtains from eq. (10)

$$\mathcal{H} = -\frac{i\hbar}{2m}\frac{\partial\pi}{\partial x}\frac{\partial\psi}{\partial x} - \frac{i}{\hbar}\pi\int K(x,y)\ \psi(y)\ dy \tag{VI.11}$$

We now show that eqs. (5,6) follow from the Hamilton's canonical equations

$$\dot{\psi} = \frac{\not\partial H}{\not\partial\pi} \tag{VI.12}$$

$$\dot{\pi} = -\frac{\not\partial H}{\not\partial\psi}\ , \tag{VI.13}$$

where

$$H(t) = \int \mathcal{H}(x,t)\,dx \tag{VI.14}$$

is the total Hamiltonian of the system, and "slash" denotes functional derivative. The functional dependence of $\mathcal{H}$ is, when varying H with respect to π,

$$\mathcal{H} = \mathcal{H}\left(\pi, \frac{\partial\pi}{\partial x}\right) \quad ,$$

so that the usual relation

$$\frac{\not{\partial} H}{\not{\partial}\pi} = \frac{\partial\mathcal{H}}{\partial\,\pi} - \frac{\partial}{\partial x} \cdot \frac{\partial\mathcal{H}}{\partial\left(\frac{\partial\pi}{\partial x}\right)} \tag{VI.15}$$

must hold. With an argument similar to that used in deriving eq. (6), we can show that the relation

$$\frac{\not{\partial} H}{\not{\partial}\psi} = \frac{\partial\mathcal{H}}{\partial\psi} - \frac{\partial}{\partial x} \cdot \frac{\partial\mathcal{H}}{\partial\left(\frac{\partial\psi}{\partial x}\right)} \tag{VI.16}$$

also holds.

The first of the Hamilton's canonical equations, eq. (12), now reads

$$\begin{aligned}
\dot{\psi} &= \frac{\not{\partial} H}{\not{\partial}\pi} \\
&= \frac{\partial\mathcal{H}}{\partial\pi} - \frac{\partial}{\partial x} \quad \frac{\partial\mathcal{H}}{\partial\left(\frac{\partial\pi}{\partial x}\right)} \\
&= -\frac{i}{\hbar}\int K(x,y)\psi(y)\,dy - \frac{\partial}{\partial x} \cdot \left(-\frac{i\hbar}{2m}\frac{\partial\psi}{\partial x}\right)
\end{aligned}$$

which is easily seen to be identical with eq. (5).

Likewise, the remaining canonical equation gives rise to the conjugate Schrödinger equation.

VI.3 We are now in a position to apply the technique of second quantization. Following the usual rule, we impose the equal-time commutation relation

$$[\psi(x), \pi(x')] = i\hbar\,\delta(x-x') \quad .$$

Since $\pi = i\hbar\psi^\dagger$, where $\psi^\dagger$ denotes the Hermitian conjugate of the operator ψ, we obtain

$$[\psi(x), \psi^\dagger(x')] = \delta(x-x') \tag{VI.17}$$

The equation of motion of an operator A with no explicit dependence on time is taken to be

$$i\hbar\dot{A} = [A,H] \tag{VI.18}$$

In particular,

$$i\hbar\dot{\psi} = [\psi,H] \quad .$$

Making use of eqs. (11,14), we find

$$\begin{aligned} i\hbar\dot{\psi} = {} & [\psi,\int_{-\infty}^{\infty} \frac{\hbar^2}{2m} \frac{\partial\psi^\dagger}{\partial x'} \frac{\partial\psi}{\partial x'}\, dx'] \; + \\ & [\psi, \iint K(x',x'')\psi^\dagger(x')\, \psi(x'')\, dx'dx''] \end{aligned} \tag{VI.19}$$

To evaluate the first commutator, we simplify the integral:

$$\int_{-\infty}^{\infty} \frac{\hbar^2}{2m} \frac{\partial\psi^\dagger}{\partial x'} \frac{\partial\psi}{\partial x'} \, dx' = \frac{\hbar^2}{2m} \psi^\dagger \frac{\partial\psi}{\partial x'} \Big|_{-\infty}^{\infty} - \frac{\hbar^2}{2m} \int \psi^\dagger \frac{\partial^2\psi}{\partial x'^2} \, dx'$$

$$= - \frac{\hbar^2}{2m} \int \psi^\dagger \frac{\partial^2 \psi}{\partial x'^2} \, dx' \quad .$$

Hence the first commutator becomes

$$[\psi, \int \frac{\hbar^2}{2m} \frac{\partial\psi^\dagger}{\partial x'} \frac{\partial \psi}{\partial x'} \, dx'] = - \frac{\hbar^2}{2m} [\psi \, , \int \psi^\dagger \frac{\partial^2\psi}{\partial x'^2} \, dx']$$

$$= - \frac{\hbar^2}{2m} \int [\psi \; (x), \psi^\dagger(x')] \frac{\partial^2\psi}{\partial x'^2} \, dx'$$

$$= - \frac{\hbar^2}{2m} \int \delta(x-x') \frac{\partial^2\psi}{\partial x'} \, dx'$$

$$= - \frac{\hbar^2}{2m} \cdot \frac{\partial^2\psi}{\partial x^2}$$

The second commutator is

$$\iint K(x',x") \, \{\psi(x)\psi^\dagger(x')\psi(x") - \psi^\dagger(x')\psi(x")\psi(x)\} \, dx'dx"$$

$$= \iint K(x',x") \, \{\psi^\dagger(x')\psi(x)\psi(x") + \delta(x-x')\psi(x") - \psi^\dagger(x')\psi(x")\psi(x)\} \, dx'dx"$$

$$= \iint K(x',x") \;\; \{ \, \delta(x-x') \; \psi(x") \, \} \, dx' \, dx"$$

$$= \int \; K(x,x") \; \psi(x") \; dx" \quad .$$

Hence

$$i\hbar\dot{\psi} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + \int K(x,x')\psi(x')\,dx' ,$$

which is formally identical with eq. (5).

The Hermitian operator

$$N = \int \psi^\dagger\psi \, dx \tag{VI.19}$$

is commonly assumed to represent the number of particles in the field. Its time derivative is given by

$$\begin{aligned} i\hbar\dot{N} &= [N,H] \\ &= \left[\int\psi^\dagger\psi dx, \int \frac{\hbar^2}{2m}\frac{\partial\psi^\dagger}{\partial x'}\frac{\partial\psi}{\partial x'}\,dx'\right] + \Big[\int\psi^\dagger\psi\,dx, \ldots \\ &\qquad \iint K(x',x'')\psi^\dagger(x')\psi(x'')dx'dx''\Big] \end{aligned}$$

The second commutator is

$$\begin{aligned} &= \iiint dx\,dx'dx''\left\{\psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x'') - \psi^\dagger(x')\psi(x'')\psi^\dagger(x)\psi(x)\right\}K(x',x'') \\ &= \iiint dx\,dx'\,dx''\Big\{\psi^\dagger(x)\left(\psi^\dagger(x')\psi(x)+\delta(x-x')\right)\psi(x'') - \\ &\qquad \psi^\dagger(x')\left(\psi^\dagger(x)\psi(x'')+\delta(x-x'')\right)\psi(x)\Big\}\cdot K(x',x'' \\ &= \iiint dx\,dx'\,dx''\left\{\psi^\dagger(x)\psi(x'')\delta(x-x') - \psi^\dagger(x')\psi(x)\delta(x-x'')\right\}K(x',x'') \\ &= \iiint dx'\,dx''\,\psi^\dagger(x')\psi(x'')\,K(x',x'') - \int dx'\,dx''\,\psi^\dagger(x')\psi(x'')\,K(x',x'') \\ &= 0 \end{aligned}$$

A similar calculation shows that the first commutator also vanishes. Hence

$$\dot{N} = 0 \; ; \qquad \text{(VI.20)}$$

therefore N is a constant of motion.

We see that the essential relations appearing in a quantized Schrödinger wave field for local potentials can be generalized to include nonlocal interactions.

BIBLIOGRAPHY

1. J.Wheeler, Phys.Rev. 50 643 (1936).

2. H.A.Bethe and J.Goldstone, Proc. Roy. Soc. A, 238 551 (1956). H.Feshbach, Ann. Rev. Nuc. Sci. 8, 49 (1958). A. Sugio, Prog. Theor. Phys. (Japan) 21, 681 (1959). M. Namiki, Prog. Theor. Phys. (Japan) 23, 629 (1960).

3. F. Blorklund and S Fernbach, Phys. Rev. 109, 1295 (1958).

4. F.Perey and B.Buck, Nuc Phys. 32, 353 (1962).

5. K.A.Brueckner, in Proceedings of the International Conference on Nuclear Structure (1960)

6. J.L.Gammel and R.M.Thaler, in Proceedings of the Seventh Annual Rochester Conference on High-Energy Nuclear Physics (1957).

7. H. Feshbach, Ann. Phys. (N.Y.) 5, 357 (1958).

8. H. Yukawa, Phys. Rev. 77, 219 (1950); 80, 1047 (1950).

9. P.A.M.Dirac, Proc. Camb. Phil. Soc. 26, 376 (1930).

10. T.Teichman and E.P.Wigner, Phys. Rev. 87, 123 (1952). P.I.Kapur and R.Peierls, Proc. Roy. Soc. A166, 277 (1937).

11. G.E.Brown, and C.T.DeDominicis, Proc. Phys. Soc. (London) A72, 70 (1958).

12. N.Austern, Phys. Rev. 137, B752 (1965).

13. E.P.Wigner, Phys. Rev. 98,145 (1955).

14. M. Razavy, to be published.

15. E.Corinaldesi and S.Zienau, Proc. Camb. Phil. Soc. 52 599 (1956).

16. G.Breit et al, Phys.Rev. 120,2227 (1960).

17. D.A.Giltinan and R.M. Thaler, Phys. Rev. 131,805 (1963).

18. See, for example, L. Schiff, Quantum Mechanics, second edition, Chap XIII.

19. S. N. Gupta, Phys. Rev. 124 (1961) 1638.

20. V. de la Cruz, B. Orman, and M. Razavy, to be published.

21. See, for example, S.S. Schweber, An Introduction to Relativistic Quantum Field Theory, Chapter 3.

B29835